A Third order Trigonometrically Fitted Modified Three-stage Runge-Kutta Method for the Numerical Solution of Periodic IVPs

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Abstract:
A new explicit trigonometrically-fitted modified Runge-Kutta method (TFMRK) is developed for the numerical integration of first-order initial value problems (IVPs) with oscillatory solutions. The newly developed method was made according to the method of Runge-Kutta Dormand and to the third algebraic order. Linear stability of the new method were examined. Numerical results are reported to show the robustness and competence of the new method compared with other existing Runge-Kutta (RK) methods.
Keywords: Runge-Kutta Methods, Trigonometrically-Fitting, Periodic Solutions, Ordinary Differential Equations, Initial Value Problems.

Introduction

Consider the numerical solution of the IVPs for the first order Ordinary Differential Equations (ODEs) in the form:

$$y' = f(x, y), \qquad y(x_0) = y_0 \qquad (1)$$

whose solutions show a periodically behavior. these problems exist in several fields of applied sciences, for instance, orbital mechanics, astrophysics, molecular dynamics, electronics and mechanics, etc. In general, most problems with oscillatory or periodically behavior are second order or higher order. Thus, it is necessary to reduce the higher order problems to first-order problems in order to solve the ODEs (1). Researchers have developed integrates with frequency-dependent coefficients using some techniques including exponential\trigonometrically-fitting (see [11, 15, 16]). Moreover, the early construction of these techniques is introduced by (see [14] and [12]). Meanwhile, Anastassi, and Simos [1] constructed two trigonometrically-fitted methods based on a classical fifth algebraic order England's Runge-Kutta method for the integration of the radial $Schrödinger$ equation which have energy with lower powers in the local truncation error. Subsequently, Bettis [3] developed a three-stage method and a four stage method, which solve the equation $y' = iwy$ without truncation error. Furthermore, Berghe et al [2] constructed exponentially fitted RK methods with s stages. A few years later, Fawzi et al. in their papers , in their papers [9] and [10] developed trigonometrically-fitted modified RK method from the fourth order to determine the approximate solution of the first-order IVPs with oscillatory solution respectively.

Meanwhile , Demba et al. in [6] and [5], developed a four-stage fourth order and four- stage fifth-order explicit trigonometrically-fitted Nystrom RKN method respectively for the numerical solution of second-order initial value problems with oscillatory solutions based on Simos RKN method. Then, Fang et al.[8] and Chen et al.[4] derived two fourth order and three practical exponentially-fitted TDRK methods respectively. Zhang et al.[17] proposed a new fifth order trigonometrically-TDRK method for the numerical solution of the radial $Schrödinger$ equation and oscillatory problems.

The remaining part of this paper is designed as follows: section 1 deals with the derivation of the proposed method. The stability property of the new method is analysed in section 2. In section 3, we present the numerical results and the last section deals with the conclusion.

Derivation of the new methods

In this part, Three-stage third order modified Runge-Kutta (RK) method using Simos technique will be derived.

Consider the three-stage explicit modified RK method given by:

$$y_{n+1} = y_n + h\sum_{i=1}^{3} b_i f(x_n + c_i h, Y_i) \quad (2)$$

$$Y_i = g_i y_n + h\sum_{j=1}^{3} a_{ij} f\left(x_n + c_i h, Y_j\right) \quad (3)$$

For g_1=1, i=1,2,3. The coefficients of the method can be expressed by the Butcher table

$$\begin{array}{c|c|ccc} 0 & 1 & & & \\ c_2 & g_2 & a_{21} & & \\ c_3 & g_3 & a_{31} & a_{32} & \\ \hline & & b_1 & b_2 & b_3 \end{array}$$

In this study, the three-stage third order RK method will be considered as given in [4]. The coefficients of the method are given in Table 1 below:

Table 1. Butcher Table for third stage third order RK method

$$\begin{array}{c|ccc} 0 & 1 & & \\ 1 & \frac{1}{2} & & \\ \frac{2}{3} & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Applying an explicit modified RK method (2) and (3) to $y' = -iwy$, we have:

$$y_{n+1} = y_n + h\sum_{i=1}^{3} b_i(-iwY_i) \quad (4)$$

$$Y_i = g_i y_n + h\sum_{j=1}^{3} a_{ij}\, Y_j \quad (5)$$

where

$$Y_1 = g_1 y_n \quad (6)$$

$$Y_2 = g_2 y_n - hwa_{21}\, Y_1 \quad (7)$$

$$Y_3 = g_3 y_n - hwa_{21}\, Y_1 - hwa_{32}\, Y_2 \quad (8)$$

Now, let $y_n = e^{iwx_n}$, computing the value of y_n and substituting to the equations (4)-(8) and by using $e^v = cos(v) + i\,sin(v)$ and comparing the real and imaginary parts, we get the following system of equations:

$$cos(v) = 1 - v^2 \sum_{i=1}^{3} b_i \sum_{j=1}^{2} a_{ij}(Y_i e^{-iwx_n}) \qquad (9)$$

$$sin(v) = v \sum_{i=1}^{3} b_i g_i \qquad (10)$$

where $v = iw$.

Solving Eqs (9) and (10) using the coefficients of the method in Table 1 for $a_{31}, a_{32}, c_2, c_3, b_1, b_2$ and b_3, we obtain the solution as given in Eqs (11) and (12):

$$a_{21} = \frac{6\,cos(v) - 6 - 2v^2 + 3\,sin(v)\,v}{v^2(4 + v^2)} \qquad (11)$$

$$g_2 = -\frac{1}{2}\frac{(6\,cos(v)\,v - 2v - 12\,sin(v) - v^3)}{(4 + v^2)v} \qquad (12)$$

which lead to our new method TFRK3.

The corresponding Taylor series expansion of the solution is given in Eqs (13) and (14):

$$a_{21} = \frac{1}{2} - \frac{1}{16}v^2 + \frac{11}{960}v^4 - \frac{37}{13440}v^6 + \frac{1661}{2419200}v^8 - \frac{6851}{39916800}v^{10} + \frac{1246879}{29059430400}v^{12} + \cdots \qquad (13)$$

$$g_2 = 1 - \frac{3}{160}v^4 + \frac{73}{13440}v^6 - \frac{83}{60480}v^8 + \frac{18269}{53222400}v^{10} - \frac{50893}{593049600}v^{12} + \cdots \qquad (14)$$

TFMRK3 method will reduce to its original method that is RK3 method as $v \rightarrow 0$. The results of different parameter were compared and analyzed to identify which problem that has contributed the smallest error.

Stability analysis

According to the stability and periodicity theory proposed by Lambert and Watson (see [13]), the stability of the new method can be analysed by the test equation:

$$y' = iwy \qquad (15)$$

and produced

$$y_{n+1} = y_n + \hat{h}BY \qquad (16)$$

$$Y = y_n G + \hat{h}AY \qquad (17)$$

where $Y = [Y_1, Y_2, \dots, Y_s]$, $G = [g_1, g_2, \dots, g_s]$, $B = [b_1, b_2, \dots, b_s]^T$ and $\hat{h} = hw$

From (17), we have:

$$Y = \left(I - \hat{h}A\right)^{-1} y_n G \qquad (18)$$

Substituting Equation (18) into Equation (16), we obtained:

$$y_{n+1} = R(\hat{h})y_n$$

where $R(\hat{h})$ represents the stability function of this new method such that:

$$R(\hat{h}) = 1 + \hat{h}B(1 - \hat{h}A)^{-1}G \qquad (19)$$

Using this new method, stability polynomial was obtained by three different stages of solution. Firstly, the value of a_{21} and g_2 were taken up to h^6 of their series solution

$$a_{21} = \frac{1}{2} - \frac{1}{16}v^2 + \frac{11}{960}v^4 - \frac{37}{13440}v^6, \qquad g_2 = 1 - \frac{3}{160}v^4 + \frac{73}{13440}v^6$$

$$R(\hat{h}) = \frac{2}{3} + \left(\frac{1}{3}\right)\hat{h} - \left(\frac{1}{48}\right)\hat{h}^3 - \left(\frac{7}{2880}\right)\hat{h}^5 + \left(\frac{1}{1120}\right)\hat{h}^7 + \left(\frac{1}{6}\right)\hat{h}^2 - \left(\frac{13}{480}\right)\hat{h}^4 + \left(\frac{227}{40320}\right)\hat{h}^6 - \left(\frac{37}{40320}\right)\hat{h}^8 \qquad (20)$$

Secondly, the value of a_{21} and g_2 were taken up to h^8 from their series solution.

$$a_{21} = \frac{1}{2} - \frac{1}{16}v^2 + \frac{11}{960}v^4 - \frac{37}{13440}v^6 + \frac{1661}{2419200}v^8$$

$$g_2 = 1 - \frac{3}{160}v^4 + \frac{73}{13440}v^6 - \frac{83}{60480}v^8$$

The stability polynomial of the new method is given as follow:

$$R(\hat{h}) = \frac{2}{3} + \left(\frac{1}{3}\right)\hat{h} - \left(\frac{1}{48}\right)\hat{h}^3 - \left(\frac{7}{2880}\right)\hat{h}^5 + \left(\frac{1}{1120}\right)\hat{h}^7 - \left(\frac{79}{345600}\right)\hat{h}^9 + \left(\frac{1}{6}\right)\hat{h}^2 - \left(\frac{13}{480}\right)\hat{h}^4 + \left(\frac{227}{40320}\right)\hat{h}^6 - \left(\frac{499}{362880}\right)\hat{h}^8 + \left(\frac{1661}{7257600}\right)\hat{h}^{10} \qquad (21)$$

Lastly, the value of a_{21} and g_2 were taken up to h^{10} of their series solution.

$$a_{21} = \frac{1}{2} - \frac{1}{16}v^2 + \frac{11}{960}v^4 - \frac{37}{13440}v^6 + \frac{1661}{2419200}v^8 - \frac{6851}{39916800}v^{10}$$

$$g_2 = 1 - \frac{3}{160}v^4 + \frac{73}{13440}v^6 - \frac{83}{60480}v^8 + \frac{18269}{53222400}v^{10}$$

The stability polynomial of the new method is given as follow:

$$R(\hat{h}) = \frac{2}{3} + \left(\frac{1}{3}\right)\hat{h} - \left(\frac{1}{48}\right)\hat{h}^3 - \left(\frac{7}{2880}\right)\hat{h}^5 + \left(\frac{1}{1120}\right)\hat{h}^7 - \left(\frac{79}{345600}\right)\hat{h}^9 + \left(\frac{27403}{479001600}\right)\hat{h}^{11} + \left(\frac{1}{6}\right)\hat{h}^2 - \left(\frac{13}{480}\right)\hat{h}^4 + \left(\frac{227}{40320}\right)\hat{h}^6 - \left(\frac{499}{362880}\right)\hat{h}^8 + \left(\frac{54811}{159667200}\right)\hat{h}^{10} - \left(\frac{6851}{119750400}\right)\hat{h}^{12} \qquad (22)$$

Stability region of the new method obtained by using the formula of Euler to equate the above three stability polynomials and solving h using the maple package.

$$R(\hat{h}) = e^{I\theta} = cos(\theta) + isin(\theta)$$

The stability interval of the new method is approximately $(-2.429,0)$ while the region can be seen in Figure 1.

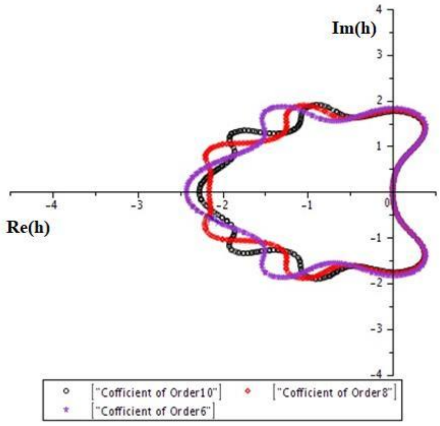

Figure 1. The stability region of TFMRK3 using various orders.

Numerical experiments

In order to study the efficiency of the developed method, we present some numerical experiments for the following three problems. The criterion used in the numerical comparisons is the decimal logarithm of the maximum global error versus the computational effort measured by the number of function evaluations required by each method and the methods used in the comparison are denoted by:

- TFMRK3: The three-stage third-order trigonometrically-fitted modified RK method derived in this paper.
- RK3: The three-stage third-order RK method given in [7].
- RK4(7): The seven-stage fourth-order RK method given in [9].
- RK5(6): The seven-stage fourth-order RK method given in [13].

The accuracy strategy used is finding log_{10} of the maximum Global error :

$$\text{The Maximum Global Error} = log_{10} \ max\|y(x_n) - y_n\|$$

where $x_n = x_0 + nh, n = 1,2,3, \dots . \frac{T-x_0}{h}$.

Problem 1. (Harmonic Oscillator [6])

$$y_1' = y_2 \qquad y_1(0) = 1$$
$$y_2' = -64y_1 \qquad y_2(0) = -2$$

with exact solution

$$y_1(x) = -\frac{1}{4}sin(8x) + cos(8x),$$
$$y_2(x) = -2\,cos(8x) - 8sin(8x)$$

Problem 2. (Inhomogeneous problem [12])

$$y_1' = y_2 \qquad y_1(0) = 1$$
$$y_2' = -100y_1 + 99sin(x) \qquad y_2(0) = 11$$

with exact solution

$$y_1(x) = cos(10x) + sin(10x) + sin(x),$$
$$y_2(x) = -10\,sin(x) + 10cos(10x) + cos(x)$$

Problem 3. (An almost Periodic Orbit problem [9])

$$y_1' = y_2, \qquad y_1(0) = 1$$
$$y_2' = -y_1 + 0.001\,cos(x), \qquad y_2(0) = 1$$
$$y_3' = y_4\,, \qquad y_3(0) = 0$$
$$y_4' = -y_3 + 0.001\,sin(x), \qquad y_4(0) = 0.995$$

with exact solution

$$y_1(x) = cos(x) + 0.0005x\,sin(x)$$
$$y_2(x) = -\,sin(x) + 0.0005x\,cos(x) + 0.0005\,sin(x)$$
$$y_3(x) = sin(x) - 0.0005x\,cos(x)$$
$$y_4(x) = cos(x) + 0.0005x\,sin(x) - 0.0005\,cos(x)$$

The efficiency of the method developed to presented in Figures 2-4 by plotting the graph of the decimal logarithm of the maximum global error against the logarithms number of function evaluations.

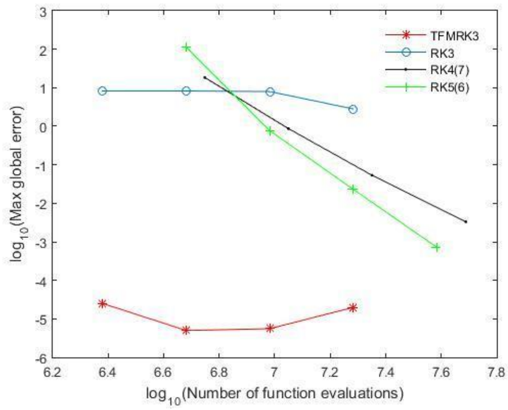

Figure 2. Efficiency curves for all methods using problem 1 with h = 0.00625, 0.0125, 0.025 and 0.05 for b = 10000.

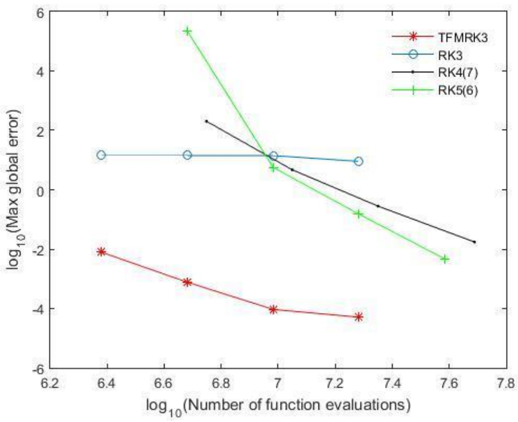

Figure 3. Efficiency curves for all methods using problem 1 with h = 0.00625, 0.0125, 0.025 and 0.05 for b = 10000.

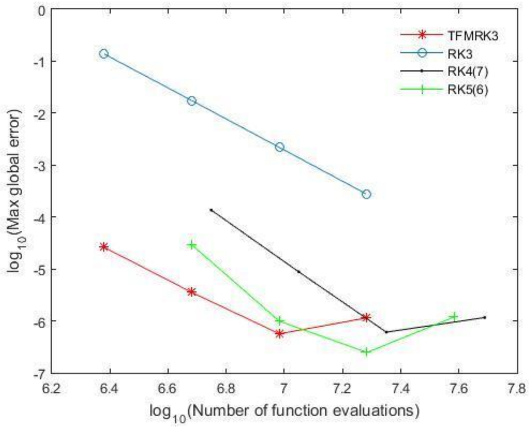

Figure 4. Efficiency curves for all methods using problem 1 with h = 0.00625, 0.0125, 0.025 and 0.05 for b = 10000.

CONCLUSION

This study has presented a newly developed trigonometrically-fitted modified Runge-Kutta method (TFMRK) in solving first-order ordinary differential equations with periodic solutions. We also analysed the linear stability of the new method. The numerical results obtained show clearly that the global error of the new method is smaller than that of the other existing methods. The proposed method is much more efficient than the other existing methods.

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